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FLOW OF GAS IN THE NEIGHBORHOOD OF AN AXISYMMETRIC VERTEX*

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The asymptotics of an axisymmetric flow near the vertex of a sharp conical body is studied. Two classes of problems are considered; a supersonic flow with attached shock wave, and a flow with a stagnation point at the vertex independent of the behavior of the flow. In the first case a linearization of the type $f = f_0 + r^\mu f_1$ relative to the purely conical flow leads to a some eigenvalue problem. Certain properties of the solutions are described. It is shown that the smallest eigenvalue is equal to zero for the largest possible (for a given M_∞) value of the cone half-angle β. In the second case an analogous linearization (relative to the state at rest) yields a spectral problem for the Legendre equation with condition that the solution is bounded. A relation describing the dependence of the smallest eigenvalue μ on β is given. The value $\beta = \beta_{(1)} = 1.138$, in particular, is found, for which $\mu = 1$. When $\beta < \beta_{(1)}$ $\partial P/\partial r = \infty$, at the cone vertex, while when $\beta > \beta_{(1)}$ $\partial P/\partial r = 0$. A comparison is drawn between the asymptotics obtained and the results of solving numerically the problem of supersonic flow past finite pointed bodies with departed shock wave.

1. Let a pointed axisymmetric body be streamlined by a steady homogeneous supersonic flow of inviscid, non-heat conducting real gas. The sharp end of the point represents a straight circular cone the axis of which is directed along the oncoming flow, and the shock wave is attached. Let us investigate the flow near the apex. The corresponding plane problem was dealt with in /1/. We shall use a polar (r, φ) coordinate system with the center at the apex. The angle φ is counted from the cone axis. Let X be the column vector of the gasdynamic quantities u, v, P and ρ where u and v are the r- and φ-velocity components, P is pressure and ρ is density. Let us denote by β the half-angle of the cone, and by α the angle if inclination of the shock wave to the axis at the vertex (Fig.1).

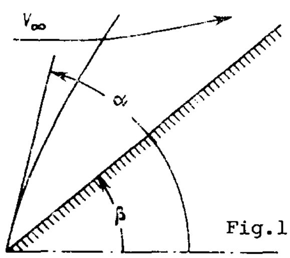

Fig.1

The solution satisfies the following system of gasdynamic equations:

$$A(X)\frac{\partial X}{\partial r} + \frac{1}{r}B(X)\frac{\partial X}{\partial \varphi} + \frac{1}{r}C(X, \varphi) = 0 \tag{1.1}$$

(the concrete form of the matrix coefficients is, so far, not important), and the boundary conditions at the shock wave and at the body. We write the solution near the vertex in the form

$$X(r, \varphi) = X_0(\varphi) + X_1(r, \varphi) \tag{1.2}$$

where X_0 describes an unperturbed motion consisting of a flow past an infinite cone with attached shock wave and $X_1 \to 0$ as $r \to 0$. Thus we have

$$B(X_0)\, dX_0 / d\varphi + C(X_0, \varphi) = 0 \tag{1.3}$$

Substituting (1.2) into (1.1) and neglecting the terms of higher order of smallness, we arrive at the homogeneous linear system

$$A(X_0)\frac{\partial X_1}{\partial r} + \frac{1}{r}B(X_0)\frac{\partial X_1}{\partial \varphi} + \frac{1}{r}D(X_0, \varphi)X_1 = 0 \tag{1.4}$$

The conical flow can be assumed given, therefore the coefficients in (1.4) are known functions of φ.

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Let us fix the value of the Mach number of the incident flow (M_∞) and choose the angle α as the parameter of the problem. When α increases from its minimum value equal to arcsin (M_∞^{-1}), a moment occurs $\alpha = \alpha^*$ when the velocity in the conical flow becomes subsonic. This takes place at the cone surface $M_0(\beta) = 1$. Further increase in α leads to widening of the subsonic zone. At $\alpha = \alpha_*$ the zone arrives at the shock wave, i.e. $M_0(\alpha) = 1$. When $\alpha > \alpha_*$, the conical flow is purely subsonic.

Next we shall follow the behavior of the quantities θ_0 and β (θ_0 denotes the angle of turn by which the flow changes its direction on passing through the discontinuity). Both angles depend on α in a nonmonotonous manner. The angle θ_0 attains its maximum first (when $\alpha = \alpha_{**} > \alpha_*$), and then the angle β (when $\alpha = \alpha^{**}$). Further increase in α is accompanied by decrease on the value of both angles, and they both vanish at the limits of variation of the angle α. We note that in the plane case there are only two "critical" angles, since $\alpha^* = \alpha_*$ and $\alpha^{**} = \alpha_{**}$. In what follows, we shall only consider the subsonic (partly or completely) range $\alpha > \alpha^*$ of the fundamental conical flow.

Let us separate the variables in (1.4): $X_1(r, \varphi) = f(r)\, Y(\varphi)$ where f is a scalar. We obtain

$$A(X_0)\, Y(\varphi) \frac{df}{dr} + B(X_0) \frac{dY}{d\varphi} \frac{f}{r} + D(X_0, \varphi)\, Y \frac{f}{r} = 0$$

From this it follows that $rf^{-1}df/dr = \mu$ is a constant, i.e. $X_1(r, \varphi) = r^\mu Y(\varphi)$ and

$$A^{-1}(BdY/d\varphi + DY) = -\mu Y \tag{1.5}$$

To obtain the boundary conditions we take the polar equation of the shock wave in the form $\varphi = \alpha + \Delta(r)$. Let ψ denote the inclination of the wave to the cone axis. Then

$$\psi = \varphi + \operatorname{arctg}\left(r \frac{d\Delta}{dr}\right) \sim \alpha + \frac{d}{dr}(r\Delta)$$

The value of $X(r, \varphi)$ behind the shock wave is determined by its inclination $X(r, \varphi) = F(\psi)$, therefore we have

$$X_0(\alpha) + \frac{dX_0(\alpha)}{d\varphi} \Delta(r) + X_1(r, \alpha) + \frac{\partial X_1(r, \alpha)}{\partial \varphi} \Delta(r) =$$
$$F(\alpha) + \frac{dF(\alpha)}{d\psi} \frac{d(r\Delta)}{dr}$$

Since $X_0(\alpha) = F(\alpha)$ and $\frac{\partial X_1(r, \alpha)}{\partial \varphi} \Delta(r) = o(X_1(r, \alpha))$, we have

$$X_1(r, \alpha) = -\frac{dX_0(\alpha)}{d\varphi} \Delta(r) + \frac{dF(\alpha)}{d\psi} \frac{d(r\Delta)}{dr} \tag{1.6}$$

Dividing (1.6) by Δ and replacing X_1 by $r^\mu Y(\varphi)$, we obtain

$$\frac{r^\mu}{\Delta(r)} Y(\alpha) = -\frac{dX_0(\alpha)}{d\varphi} + \frac{dF(\alpha)}{d\psi} \frac{1}{\Delta(r)} \frac{d(r\Delta)}{dr}$$

therefore $\Delta = r^\mu$ and we have

$$Y(\alpha) = -\frac{dX_0(\alpha)}{d\varphi} + (1 + \mu) \frac{dF(\alpha)}{d\psi} \tag{1.7}$$

The parameter μ can be eliminated from the four relations (1.7), and this yields three linear boundary conditions at $\varphi = \alpha$. At the other boundary $(\varphi = \beta)$ we have a single condition $v_1(\beta) = 0$, and we obtain the traditional eigenvalue problem in the class of bounded functions (only the positive values of μ relate to the initial problem). The basic question is that of the manner in which the eigenvalues μ and eigenfunctions $Y_\mu(\varphi)$ depend on α? We note that the negative indices have a definite sensé; they correspond to the asymptotics as $r \to \infty$ for a body conical at sufficiently large r. In this case the eigenfunctions may be (and in fact are) unbounded at the body. Therefore the condition $v_1(\beta) = 0$ must be replaced by $\lim_{\varphi \to \beta} (v_1 u_0 - v_0 u_1) = 0$.

Let us formulate two assertions.

1. The spectral problem under consideration has, at $\alpha > \alpha^*$, a simple discrete spectrum $\mu_m(\alpha) \geqslant 0$ $(m = 0, 1, \ldots)$ "generated at infinity" when $\alpha = \alpha^*$, i.e. for all m $\mu_m(\alpha^*) = \infty$.

2. The function $\mu_0(\alpha)$ is defined right up to $\alpha = \alpha^{**}$; $\mu_0(\alpha^{**}) = 0$. The remaining $\mu_m(\alpha) > 0$.

The first assertion is accepted without proof, be referring to the numerical results and to analogy with the plane case. Variability of the coefficients and the fact that the subsonic region occupies, in the axisymmetric case, only a part of the space between the body and the shock wave, should not significantly influence the spectral properties.

The second assertion needs proving and elucidation, since the analogous result for the plane flows cannot be transferred automatically; the angle α^{**} (plane angle) "splits" into α^{**} and α_{**}. So, let $Y(\varphi)$ be the solution for $\mu = 0$. Then by virtue of (1.5) we have

$$B\,dY/d\varphi + DY = 0 \tag{1.8}$$

Therefore $Y(\varphi)$ is a solution of the system (1.1) independent of r, i.e. the solution represents a conical perturbation of the conical solution. But such a perturbation can only be caused by variation in the value of the angle α of inclination of the wave. Therefore $Y = dX_0(\varphi)/d\alpha$ (all gasdynamic functions depend parametrically on α). In particular, at the cone surface we have

$$0 = v_1(\beta) = \frac{dv_0(\beta)}{d\alpha} = \frac{dv_0(\beta)}{d\beta}\,\frac{d\beta}{d\alpha} \tag{1.9}$$

When $\varphi = \beta$, we have $dv_0/d\varphi = -2u_0 \neq 0$, therefore $dv_0(\beta)/d\beta = -dv_0(\beta)/d\varphi \neq 0$. Then from (1.9) it follows that $d\beta/d\alpha = 0$, i.e. $\alpha = \alpha^{**}$ which completes the proof. Since α^{**} represents the boundary separating the "weak" and "strong" shock waves, this means that μ_0 and μ_1 is the smallest index for the "weak" and "strong" wave respectively. We also note that $\mu_m(\pi/2) = 2m - 1$.

2. We solve the spectral problem numerically using the simplest procedure, namely ranging over μ, and the Cauchy problem with initial values at the shock wave is solved for $\varphi = \alpha$ at every step. The equations and boundary conditions for determining X_0 are well known. A system of the type (1.5) for $Y_\mu(\varphi) = (u_1, v_1, P_1, \rho_1)$ has the form

$$\begin{aligned}
&v_0 u_1^{\cdot} = v_0 v_1 - \mu(u_0 u_1 + \rho_0^{-1} P_1)\\
&(c_0^2 - v_0^2)\, v_1^{\cdot} = v_0(v_0 u_1 + 2u_0 v_1) - v_0^{\cdot}(Q - 2v_0 v_1) - c_0^2(K_1 + u_1) - Q(K_0 + u_0) + \mu(u_0 v_0 v_1 - c_0^2 u_1 - u_0 \rho_0^{-1} P_1)\\
&(c_0^2 - v_0^2)\, P_1^{\cdot} = \gamma[K_0 v_0 P_1 + P_0(K_1 v_0 + K_0 v_1)] - P_0^{\cdot}(Q - 2v_0 v_1) + \mu[\gamma P_0(v_0 u_1 - u_0 v_1) + u_0 v_0 P_1]\\
&Q = \gamma\rho_0^{-1}(P_1 - P_0\rho_0^{-1}\rho_1), \quad K_0 = u_0 + v_0 \operatorname{ctg}\varphi \quad K_1 = u_1 + v_1 \operatorname{ctg}\varphi
\end{aligned} \tag{2.1}$$

and the relations at the shock wave are

$$\begin{aligned}
&v_0\rho_1 + \rho_0 v_1 = -\rho_0 v_0^{\cdot} - \rho_\infty q_\infty \cos\alpha + \mu(\rho_0 u_0 - \rho_\infty q_\infty \cos\alpha)\\
&P_1 - (\rho_\infty q_\infty \sin\alpha)\, v_1 = \rho_\infty q_\infty[(v_0 + q_\infty \sin\alpha)\cos\alpha + (u_0 + v_0^{\cdot})\sin\alpha] -\\
&\quad P_0^{\cdot} + \mu\rho_\infty q_\infty(v_0 + q_\infty \sin\alpha)\cos\alpha\\
&u_1 = -(1 + \mu)(v_0 + q_\infty \sin\alpha)
\end{aligned} \tag{2.2}$$

The systems (2.1) and (2.2) are closed by the equation

$$u_0 u_1 + v_0 v_1 + \frac{Q}{\gamma - 1} = 0 \tag{2.3}$$

which is a variant of the Bernoulli integral. The solution of the Cauchy problem (2.1) with initial values (2.2) presents no difficulties. Some care should be exercised when approaching the boundary $\varphi = \beta$, since the coefficient v_0 accompanying $u_1^{\cdot}$ vanishes at this boundary. It can be shown that every solution of the system (2.1) is bounded when $\mu \geqslant 0$, and this allows the use of ranging. We have also studied the converse problem, i.e. the value of μ_m was specified (say μ_0) and the corresponding value of α determined. This is needed e.g. for determining $\alpha^{**}(\mu_0 = 0)$ or of the Crocco point $\alpha_c(\mu_0 = 1)$. The numerical algorithm consists of ranging over α.

Table 1

M_∞	1,5	2,0	2,5	3,0	5.0	10	20
α^*	0.9535	1.0000	1,0445	1.0757	1,1304	1,1572	1.1643
β^*	0,4419	0.6388	0.7463	0.8098	0,9099	0.9557	0.9675
α_*	1,0866	1.0731	1.0934	1.1129	1.1534	1.1752	1,1812
β_*	0.5095	0.6766	0,7707	0,8279	0,9207	0,9640	0.9753
α_{**}	1.1622	1.1287	1.1307	1,1387	1.1621	1.1773	1,1817
β_{**}	0.5301	0,6971	0.7857	0.8386	0,9245	0.9649	0.9755
α_c	1.1200	1,1364	1.1616	1,1805	1.2162	1.2346	1.2396
β_c	0.5208	0.6993	0.7953	0,8523	0.9431	0,9851	0,9961
α^{**}	1.2042	1,2115	1.2296	1.2442	1.2725	1,2875	1.9916
β^{**}	0.5334	0.7102	0.8049	0.8611	0.9509	0,9924	1.0032

Table 2

α	β	μ_0	μ_1	μ_2
0.9535	0.4419	∞	∞	∞
1.0243	0.4801	6.103	>30	>30
1.0850	0.5089	1.878	12,78	23.78
1,1457	0.5272	0,586	7,665	15,04
1,2064	0,5334	—	5,328	11,12
1,2672	0.5252	—	3,932	8,705
1,3279	0,4995	—	2,975	6,975
1,3886	0,4527	—	2,269	5,639
1,4493	0,3799	—	1,727	4,566
1,5101	0,2711	—	1,306	3,692
1,5708	0,0000	—	1,000	3,000

Table 1 gives various "critical" values of the angle of inclination of the shock wave, i.e. $\alpha^*, \alpha_*, \alpha_{**}, \alpha^{**}$ and α_c , as well as the corresponding values of β. Table 2 demonstrates the dependence on α of three smallest indices μ_0, μ_1, μ_2 and the angle β for $M_\infty = 1.5$. Figs.2a and 2b depict the functions $P_1(\varphi)$ corresponding to the first three positive values of μ for a "weak" ($M_\infty = 10, \alpha = 1.281, \beta = 0.992$) and "strong" ($M_\infty = 10$, $\alpha = 1.488, \beta = 0.726$) shock waves. The numbers correspond to the numbers of the eigenfunctions.

Analysis of the results for various M_∞ makes possible the assertion that the number of zeros of the function $P_1(\varphi)$ corresponding to $\mu = \mu_m$ is m. This is also true in the plane case. For the remaining functions no such agreement with the plane flow exists, but nevertheless the number of zeros increases approximately as m. We note that the discrepancy between the solutions corresponding to the "weak" and "strong" shock waves is not so fundamental as to allow the use of local analysis in postulating the nonexistence of the flows with "strong" waves. The fact that μ_0 vanishes when $\alpha = \alpha^{**}$ was first discovered by numerical methods /2/. Some results relevant to the present problem were obtained in /3/.

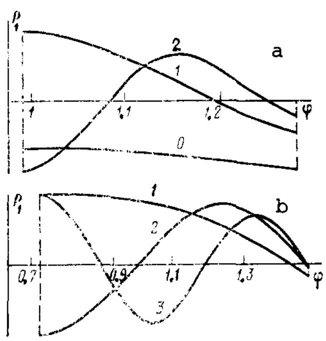

Fig.2

3. Let us consider the case $\beta > \beta^{**}$ in which the flow with attached shock wave becomes impossible. A subsonic shock layer of finite thickness is formed, and the tip of the conical vertex becomes a stagnation point. Let us investigate the flow structure near the vertex using the notation and coordinate system of Sect.1. The solution satisfies the following system of equations:

$$u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \varphi} + \frac{1}{\rho}\frac{\partial P}{\partial r} - \frac{v^2}{r} = 0 \quad u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \varphi} + \frac{1}{\rho r}\frac{\partial P}{\partial \varphi} + \frac{uv}{r} = 0 \tag{3.1}$$
$$\frac{\partial(\rho u)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v)}{\partial \varphi} + \frac{\rho}{r}(2u + v\operatorname{ctg}\varphi) = 0, \quad \frac{u^2+v^2}{2} + \frac{\gamma}{\gamma-1}\frac{P}{\rho} = B$$

We write the solution in the form

$$u = r^{\mu/2}u_1(\varphi) + \ldots; \quad v = r^{\mu/2}v_1(\varphi) + \ldots; \tag{3.2}$$
$$P = P_0 + r^\mu P_1(\varphi) + \ldots; \quad \rho = \rho_0 + r^\mu \rho_1(\varphi) + \ldots$$

where P_0 and ρ_0 are the pressure and density at the stagnation point. We restrict ourselves to the case of locally constant entropy. This assumption leads to the relation $P_1(\varphi)/P_0 = \gamma\rho_1(\varphi)/\rho_0$ which replaces one of the equations of the corresponding system. Linearizing (3.1) with help of (3.2) and eliminating v_1 , we obtain

$$u_1'' + u_1' \operatorname{ctg}\varphi + \nu(\nu+1)u_1 = 0 \qquad (\nu = \mu/2 + 1)$$

The boundary conditions are

$$u_1'(\beta) = 0 \quad (v_1(\beta) = 0), \quad u_1(\pi) < \infty$$

Carrying out the substitution $x = \cos\varphi$, we arrive at the final formulation of the eigenvalue problem. In order to stress the dependence on ν, we introduce the lower index. We have

$$(1-x^2)\frac{d^2u_\nu}{dx^2}-2x\frac{du_\nu}{dx}+\nu(\nu+1)u_\nu=0,\ \frac{du_\nu}{dx}=0,\quad x=b=\cos\beta;\quad u_\nu(-1)<\infty \tag{3.3}$$

Solution of (3.3) yields $u_\nu(x)$, and for $v_\nu(x)$ we obtain

$$v_\nu(x)=-\frac{\sqrt{1-x^2}}{\nu}\frac{du_\nu}{dx} \tag{3.4}$$

Equation (3.3) is a Legendre equation of order ν. Its general solution is

$$u_\nu(x)=C_1P_\nu(x)+C_2Q_\nu(x),\ \nu\geqslant 1,\ x\in[-1,\ b] \tag{3.5}$$

where P_ν and Q_ν are the Legendre functions of first and second kind. We use the asymptotic behavior of P_ν and Q_ν near the point $x=-1$, and demand that $u_\nu(-1)=-1$. This yeids

$$u_\nu(x)=-\cos\nu\pi P_\nu(x)+2\pi^{-1}\sin\nu\pi Q_\nu(x) \tag{3.6}$$

The condition of impermeability $u_\nu'(b)=0$ determines the eigenvalue ν as the root of the transcendental equation

$$-\cos\nu\pi\,[bP_\nu(b)-P_{\nu-1}(b)]+2\pi^{-1}\sin\nu\pi\,[bQ_\nu(b)-Q_{\nu-1}(b)]=0 \tag{3.7}$$

(here we have used the relation $(1-x^2)\,dP_\nu/dx=\nu\,(P_{\nu-1}-xP_\nu)$ and the analogous relation for Q_ν). In this manner the problem is reduced to that of solving the equation (3.7) for ν and subsequent determination of $u_\nu(x)$ and $v_\nu(x)$ in accordance with (3.6) and (3.4). We note that $\mu(\pi/2)=2,\ \mu(0)=0$, therefore $\nu(\pi/2)=2,\ \nu(0)=1$.

Table 3

$\beta°$	μ	$\beta°$	μ	$\beta°$	μ	$\beta°$	μ
90	2.000	65	0.984	40	0.394	15	0.063
85	1.753	60	0.848	35	0.308	10	0.029
80	1.532	55	0.717	30	0.231	5	0.008
75	1.334	50	0.598	25	0.165	0	0.000
70	1.155	45	0.490	20	0.109		

Let us quote some results for the first, minimal eigenvalue. Table 3 gives the quantity μ as a function of the angle β. Let us single out the value $\beta=\beta_{(1)}$, at which $\mu=1$ ($\nu=1,5$). The computations yield $\beta_{(1)}=1.138$. Thus we have $\mu>1$ for $\pi/2\geqslant\beta>\beta_{(1)}$ and the function $P(r,\varphi)$ is differentiable at the zero (and $\partial P/\partial r=0$ when $r=0$). For $\beta<\beta_{(1)}$ we have $\partial P(0,\varphi)/\partial r=\infty$ ($\mu<1$). The velocity components always have a singularity, since the corresponding index ($\mu/2$) does not exceed unity. Fig.3 depicts the graphs $u_1(\varphi)$, $v_1(\varphi)$ and $2P_1(\varphi)/\rho_0$ for $\beta=40°$.

4. The data obtained explain certain specific features of the behavior of the gasdynamic functions observed in the course of solving numerically the problem of supersonic flow past finite pointed bodies with detached shock wave. To do this we use the results of /4/, in which the authors studied a flow past a conically pointed body using a sphere as an example of such a body. The geometry of the problem is shown schematically in Fig.4, and plots of the pressure distribution $P(\theta)$ along the body are given for various bodies: $\beta=70, 60, 50, 45°$ (curves *1–4* respectively) for $M_\infty=2$ (the results for $M_\infty=6$ are similar). We note that the parameters of the oncoming flow do not affect the asymptotics, since they appear only in the normalizing multipliers. It is clear that the difference mesh chosen for the purpose of solving these problems does not furnish any details about the behavior of the functions near the sharp end. However, the boundary $\beta=\beta_{(1)}$ is seen clearly: for $\beta=45°$ and $50°$ $\partial P(0)/\partial\theta=0$ and for $\beta=70°$ $\partial P(0)/\partial\theta=\infty$. The case $\beta=60°$ is almost boundary.

The agreement between the behavior of the velocity and the asymptotics is also fully satisfactory. Thus, when $\varphi=\beta$ (at the body) the asymptotics can be well observed even when the mesh is sufficiently coarse. The situation becomes much worse at the stream line $\varphi=\pi$. Here the results of the numerical difference computations fail to disclose any singularities. This is apparently connected with the strong dependence on φ of the prolongation of the region of asymptotic behavior. In addition, the computational nodes lying on the stream line may have missed this region.

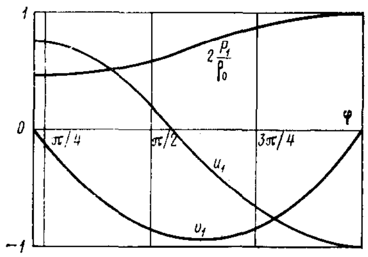

Fig.3

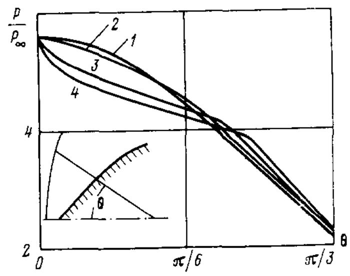

Fig.4

The asymptotics obtained may also be used in constructing numerical algorithms and in the study of boundary layer near the sharp end. Here the singularity in the pressure distribution appearing at $\beta < \beta_{(1)}$ must be taken into account.

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